STEP MATHEMATICS 2

2019

Mark Scheme

STEP II 2019 Mark Scheme

1			$f'(x) = g(x) + (x-p)g'(x)$ Tangent passes through $(a, (a-p)g(a))$ Equation of tangent is $y = \big(g(a) + (a-p)g'(a)\big)(x-a) + (a-p)g(a)$ (or equivalent equation) Substitution of $x = p$ into equation of tangent $y = -(a-p)^2 g'(a)$ Verification that if $g'(a) = 0$, then $y = 0$ If $y = 0$ then $g'(a) = 0$ because $a \neq p$	M1 M1 A1 E1 E1 E1 (AG) (6 marks)
	(i)		$g(x) = A(x-q)(x-r)$ identified $g'(a) = 0 \Rightarrow 2a = r + q$ (legitimately obtained) Gradient of tangent is $g(a) + (a-p)g'(a)$ $= A(a-q)(a-r)$ $= -\frac{1}{4}A(r-q)^2$	M1 A1 (AG) M1 A1 (4 marks)
	(ii)		By symmetry, the gradient of the second tangent is $-\frac{1}{4}A(p-q)^2$ (can be implied) Parallel iff $(p-q)^2 = (q-r)^2$ $\Leftrightarrow q - p = r - q$ since $p < q < r$. Tangent at $x = q$, $y = A(q-p)(q-r)(x-q)$, Meets curve again when $(q-p)(q-r)(x-q) = (x-p)(x-r)(x-q)$ $\Leftrightarrow (q-p)(q-r) = (x-p)(x-r)$ since $x \neq q$ (cancellation must be justified for M1, can be awarded if used correctly on $(x-q)^2(x-p-r+q)$ later) $\Leftrightarrow (x-q)(x-p-r+q) = 0$ $\Leftrightarrow x = p + r - q$ or $x = q$ Therefore there is only one point of intersection between the tangent and the curve if and only if $p + r - q = q$, which is if and only if the tangents are parallel. *One E mark for each direction.*	 B1 M1 A1 E1 M1 M1 M1 A1 E1 E1 (AG) (10 marks)

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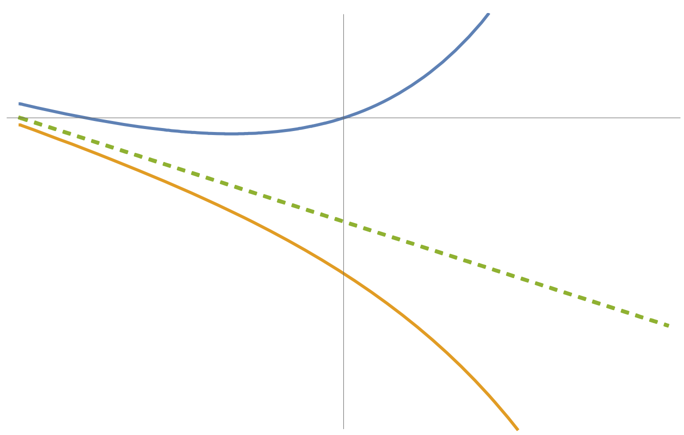
2			Sketch with areas $\int_0^x f(t)\,dt$, $\int_0^{f(x)} f^{-1}(y)\,dy$ and rectangle correctly identified. (One mark any one)	G1 G1 (2 marks)
	(i)		$g(0)(g(0)^2+1)=0$ factorised $g(0)$ real so $g(0)=0$ (must be justified) $1=(3g(t)^2+1)g'(t)$ $(3g(t)^2+1)>0$ so $g'(t)>0$ $g(2)^3+g(2)-2=0$ $(g(2)-1)(g(2)^2+g(2)+2)=0$ $\Delta=-7<0$ so $g(2)=1$ or g(2)>0 justified $g^{-1}(s)=s^3+s$ $\int_0^2 g(t)dt=2g(2)-\int_0^{g(2)} g^{-1}(s)ds$ $=\frac{5}{4}$	M1 A1 (AG) M1 A1 (AG) M1 A1 B1 M1 A1 (9 marks)
	(ii)		$h(t)=g(t+2)$ so $h(0)=g(2)=1$ and $h'(t)>0$ $(h(8)-2)(h(8)^2+2h(8)+5)=0$ $h(8)=2$ correctly justified $h^{-1}(s)=s^3+s-2$ $\int_0^8 h(t)dt+\int_{h(0)}^{h(8)} h^{-1}(s)ds=16$ (or similar correct equation) $\int_0^8 h(t)dt=16-\int_1^2(s^3+s-2)ds$ $=16-[\frac{s^4}{4}+\frac{s^2}{2}-2s]_1^2$ (integration) $=12\frac{3}{4}$	M1 A1 M1 A1 B1 M1 A1 M1 A1 (9 marks)

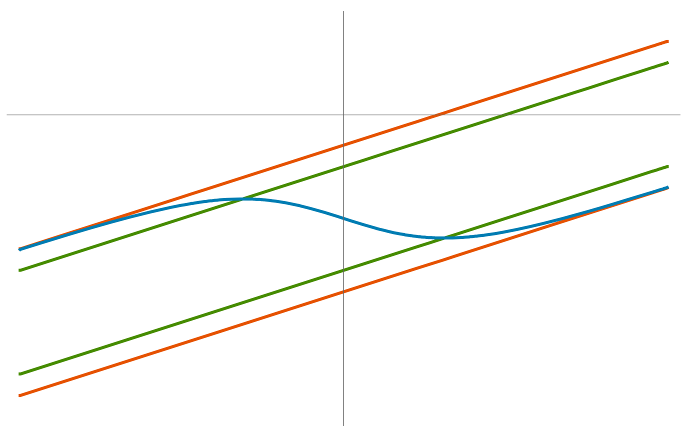
3			$\lvert x_1+x_2\rvert$ is maximised when both have the same sign, In which case $\lvert x_1+x_2\rvert = \lvert x_1\rvert + \lvert x_2\rvert$. Thus, $\lvert x_1+x_2\rvert \le \lvert x_1+x_2\rvert$ (or by consideration of all four combinations of signs separately) $\lvert x_1+\cdots+x_{n-1}+x_n\rvert \le \lvert x_1+\cdots+x_{n-1}\rvert + \lvert x_n\rvert$ $\le \cdots$ $\le \lvert x_1\rvert+\cdots+\lvert x_{n-1}\rvert+\lvert x_n\rvert$ by induction	E1 E1 (2 marks)
	(i)	(a)	$\lvert f(x)-1\rvert = \lvert a_1x+\cdots+a_{n-1}x^{n-1}+x^n\rvert$ $\le \lvert a_1x\rvert+\cdots+\lvert a_{n-1}x^{n-1}\rvert+\lvert x^n\rvert$ $= \lvert a_1\rvert\lvert x\rvert+\cdots+\lvert a_{n-1}\rvert\lvert x\rvert^{n-1}+\lvert x\rvert^n$ $\le A(\lvert x\rvert+\cdots+\lvert x\rvert^{n-1})+\lvert x\rvert^n$ $\le A(\lvert x\rvert+\cdots+\lvert x\rvert^{n-1}+\lvert x\rvert^n)$ (justified) $= A\frac{\lvert x\rvert(1-\lvert x\rvert^n)}{1-\lvert x\rvert}$ $\le A\frac{\lvert x\rvert}{1-\lvert x\rvert}$ (justified)	 M1 M1 M1 M1 M1 A1 (AG) (6 marks)
		(b)	$1 \le \frac{A\lvert\omega\rvert}{1-\lvert\omega\rvert}$ using $f(\omega)=0$ $1 \le (A+1)\lvert\omega\rvert$ (with sign of $1-\lvert\omega\rvert$ justified) $A+1 \ge 1 \ge \lvert w\rvert$	M1 A1 (AG) B1 (AG) (3 marks)
		(c)	If $\lvert\omega\rvert > 1$, $0 = \omega^n f\left(\frac{1}{\omega}\right)$ $= 1+a_{n-1}\omega+\cdots+a_1\omega^{n-1}+\omega^n$ Inequalities continue to hold since $\lvert a_i\rvert \le A$ If $\lvert\omega\rvert = 1$, then $1+A \ge 1 \ge \frac{1}{1+A}$ since $A>0$	 M1 E1 E1 (3 marks)
	(ii)		$f(x) = x^5 - x^4 - \frac{100}{135}x^3 - \frac{91}{135}x^2 - \frac{126}{135}x + 1$ Use $A=1$. Integer roots with $\frac{1}{2} \le \lvert\omega\rvert \le 2$ could only be ±1 or ±2 $f(\pm2) \ne 0$ because numerator is odd (or any valid justification) $f(1) = -\frac{182}{135} \ne 0$ $f(1) = 0$ $x=1$ is the only integer root.	B1 M1 M1 E1 A1 A1 (6 marks)

4	(i)		$\sin\frac{\pi}{9}\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ $=\frac{1}{2}\sin\frac{2\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ $=\frac{1}{8}\sin\frac{8\pi}{9}$ $=\frac{1}{8}\sin\frac{\pi}{9}$ (use of $\sin(\pi-x)=\sin(x)$) $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}$	B1 M1 M1 A1 (4 marks)
	(ii)		$\sin\left(\frac{x}{2^n}\right)\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right)$ $=\frac{1}{2}\sin\left(\frac{x}{2^{n-1}}\right)\prod_{k=0}^{n-1}\cos\left(\frac{x}{2^k}\right)$ $=\cdots$ (convincing use of induction or repeated application) $=\frac{\sin(2x)}{2^{n+1}}$ (induction end point correct) $\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}$ $\sum_{k=0}^{n}\log\left(cos\left(\frac{x}{2^k}\right)\right)=\log(\sin(2x))-\log\left(sin\left(\frac{x}{2^n}\right)\right)-\log(2^{n+1})$ $\sum_{k=0}^{n}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)=-2\cot(2x)+\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)$ (justified with differentiation)	B1 M1 E1 A1 M1 (diff) M1 (division) A1 (AG) (7 marks)
	(iii)		B1 – switch to product starting at 0 M1 – set up as limiting case of product to n M1 – apply small angle for sin A1 – correct answer $\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)\cos(x)}$ $=\frac{2\sin(x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}$ $\sim\frac{\sin(x)}{2^n\times\left(\frac{x}{2^n}\right)}$ $=\frac{\sin(x)}{x}$	 M1 M1 M1 A1 (AG)

			$\sum_{j=2}^{n} \frac{1}{2^{j-2}} \tan\left(\frac{\pi}{2^j}\right)$ $= \sum_{k=0}^{n} \frac{1}{2^k} \tan\left(\frac{\pi/4}{2^k}\right)$ $= \lim_{n\to\infty} \left(\frac{1}{2^n} \cot\left(\frac{\pi/4}{2^n}\right) - 2\cot\left(\frac{\pi}{2}\right)\right)$ $= \lim_{n\to\infty} \left(\frac{1}{2^n \tan\left(\frac{\pi/4}{2^n}\right)}\right)$ $= \lim_{n\to\infty} \left(\frac{1}{2^n \left(\frac{\pi/4}{2^n}\right)}\right)$ $= \frac{4}{\pi}$	M1 M1 M1 M1 A1 (9 marks)

5	(i)		Constant iff $a = f(a)$ $\Leftrightarrow a = p + (a-p)a$ $\Leftrightarrow 0 = (a-p)(a-1)$ $\Leftrightarrow a = p$ or $a = 1$. Period 2 $\Leftrightarrow a = f(f(a))$ $\Leftrightarrow 0 = (a-p)(-1 + 2ap - pa^2 + a^3)$ (factorisation) $\Leftrightarrow 0 = (a-p)(a-1)(a^2 + (1-p)a + 1)$ If $a = p$ or $a = 1$, then sequence is constant. The quadratic has solutions when $(p-1)^2 \geq 4$. If $(p-1)^2 > 4$, i.e. $p > 3$ or $p < -1$, the solutions are distinct. They are not both $1, p$ since the sum of the roots is $p - 1 \neq p + 1$ So for $p > 3$ or $p < -1$, one of the roots of the quadratic gives a sequence of period 2. If $p = 3$, $a = 1$ so not period 2. If $p = -1$, $a = -1 = p$ so not period 2.	M1 M1 A1 M1 M1 A1 B1 M1 E1 E1 (AG) B1 B1 (12 marks)
	(ii)		No value of a for which the sequence is constant $\Leftrightarrow f(a) = a$ has no solution $\Leftrightarrow f(x) > x$ or $f(x) < x$ for all x But $f(x) > x$ for large x. So cannot have $f(x) < x$ for all x. If no value of a for which sequence constant, then $f(x) > x$ for all x So $f\big(f(x)\big) > f(x) > x$ for all x And hence no solution to $f\big(f(a)\big) = a$. Setting $p = q$, gives (i). Then if $-1 \leq p \leq 3$, there is no period 2 sequence but a constant sequence exists.	 E1 (→) E1 (←) E1 E1 E1 E1 E1 E1 (8 marks)

6	(i)		If $y = mx + c$, Then the differential equation becomes	
			$m = mx + c + x + 1$ $m = -1, c = -2$	M1
			$y = -x - 2$	A1
			$\frac{dy}{dx} = 0 \Rightarrow y + x + 1 = 0 \Rightarrow y = -x - 1$	E1 (AG)
			$y = y_3(x)$ cannot cross the line $y = -x - 2$. So if $y_3(0) < -2$, it cannot reach the line $y = -x - 1$ and hence has no stationary points.	E1
			At a stationary point, $\frac{d^2y}{dx^2} = \frac{dy}{dx} + 1 = y + x + 2 = 1 > 0$ so minimum	E1
			$\frac{dY}{dx} = Y + 2$	M1
			$\log(Y + 2) = x + c$	M1
			$Y = -2 + Ae^x$ $y = -x - 2 + Ae^x$	A1
			$y(0) = 0 \Rightarrow A = 2$ $y(0) = -3 \Rightarrow A = -1$ (attempt at both)	M1
			So $y = -x - 2 + 2e^x$ So $y = -x - 2 - e^x$ (both)	
				
			Curves tending to asymptote to the left	G1
			Curve above line through origin tending to ∞	G1
			Curve below line tending to $-\infty$	G1
				(12 marks)
	(ii)		If $y = mx + c$, Then the differential equation becomes $m = (mx + c)^2 + 4(mx + c) + x^2 - 4x - 2x(mx + c) + 3$ $0 = (m^2 - 2x + 1)x^2 + (2mc + 4m - 4 - 2c) + c^2 + 4c + 3 - m$	
			From x^2: $m = 1$ From x: $2mc + 4m - 4 - 2c = 2c + 4 - 4 - 2c = 0$	

			From 1: $c^2 + 4c + 2 = 0 \Rightarrow c = -2 \pm \sqrt{2}$ Any of these equations Correct values of m and c Solutions: $y = x - 2 \pm \sqrt{2}$ $\frac{dy}{dx} = (y - x)^2 + 4(y - x) + 3$ (writing as a function of $y - x$) $= (y - x + 3)(y - x + 1)$ Stationary pts: $y = x - 1$ or $y = x - 3$ Between these lines the gradient is negative. (Correctly justified) So stationary points on $y = x - 1$ are maxima and stationary on $y = x - 3$ are minima.  Curve does not intersect other solutions Curve has stationary points on correct lines	 M1 A1 M1 A1 A1 A1 G1 G1 (8 marks)

7	(i)		$\boldsymbol{a}\cdot(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})=0$ $\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}=-1$ and cyclic permutations $\boldsymbol{a}\cdot\boldsymbol{b}=-\frac{1}{2}$ legitimately obtained $\cos\theta=-\frac{1}{2}$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ $\theta=120°$ Similarly, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $120°$. Justification of equilateral triangle by sketch or otherwise ABC is equilateral	M1 M1 A1 M1 A1 M1 M1 A1 (8 marks)
	(ii)		$\boldsymbol{a_1}\cdot\boldsymbol{a_2}+\boldsymbol{a_1}\cdot\boldsymbol{a_3}+\boldsymbol{a_1}\cdot\boldsymbol{a_4}=-1$ and cyclic permutations Linear combination of these equations $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=\boldsymbol{a_3}\cdot\boldsymbol{a_4}$ (legitimately obtained)	M1 M1 A1 (AG) (3 marks)
		(a)	Angles $\angle A_1OA_2=\angle A_3OA_4$ By symmetry, $\angle A_2OA_3=\angle A_4OA_1$ The $\boldsymbol{a_i}$ are distinct and unit length so no angles are zero (accept justification by sketch) $A_1A_2A_3A_4$ is a rectangle	M1 M1 A1 (3 marks)
		(b)	$(A_1A_2)^2=(\boldsymbol{a_1}-\boldsymbol{a_2})^2$ $=\boldsymbol{a_1^2}+\boldsymbol{a_1^2}-2\boldsymbol{a_1}\cdot\boldsymbol{a_2}$ $=2-2\boldsymbol{a_1}\cdot\boldsymbol{a_2}$ By symmetry, $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=\boldsymbol{a_1}\cdot\boldsymbol{a_3}=\boldsymbol{a_1}\cdot\boldsymbol{a_4}$ So $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=-\frac{1}{3}$ So $(A_1A_2)^2=\frac{8}{3}$ $A_1A_2=\frac{2\sqrt{2}}{\sqrt{3}}$	 M1 M1 M1 A1 M1 A1 (6 marks)

8	(i)		$f(\boldsymbol{M}) = f(\boldsymbol{MI}) = f(\boldsymbol{M})f(\boldsymbol{I})$ $\Rightarrow f(\boldsymbol{I}) = 1$ since $f(\boldsymbol{M}) \neq 0$	M1 A1 (AG) (2 marks)
	(ii)		$f(J)^2 = f(J^2)$ $= f(I) = 1$ $\Rightarrow f(J) = -1$ since $f(J) \neq 1$ $f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ $= -f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ legitimately obtained $f\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right) = f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ $= -f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right)$ legitimately obtained	M1 M1 A1 M1 A1 (AG) M1 A1 (AG) (7 marks)
	(iii)		Using first equality in previous part (or otherwise correctly justified) $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = -f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right)$ $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0$ $f\left(\begin{pmatrix} a & b \\ ka & kb \end{pmatrix}\right) = f\left(\boldsymbol{K}\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right)$ $= 0$	M1 M1 M1 A1 (AG) (4 marks)
	(iv)		$\boldsymbol{K}^{-1}\boldsymbol{PK} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ $f(\boldsymbol{K})f(\boldsymbol{K}^{-1}) = F(\boldsymbol{I}) = 1 \Rightarrow f(\boldsymbol{K}^{-1}) = f(\boldsymbol{K})^{-1}$ $f(\boldsymbol{K}^{-1}\boldsymbol{PK}) = f(\boldsymbol{K}^{-1})f(\boldsymbol{PK})$ (must use two stages) $= f(\boldsymbol{K}^{-1})f(\boldsymbol{P})f(\boldsymbol{K})$ $= f(\boldsymbol{P})$ $\boldsymbol{P}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $f(\boldsymbol{P}^2) = f(\boldsymbol{P}) \Rightarrow f(\boldsymbol{P}) = 0$ or 1 $\boldsymbol{P}^{-1}$ exists so $f(\boldsymbol{P})f(\boldsymbol{P}^{-1}) = 1 \Rightarrow f(\boldsymbol{P}) \neq 0$	B1 M1 M1 A1 (AG)? B1 M1 A1 (7 marks)

9	(i)		$\boldsymbol{r} = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix}$	M1
			$\boldsymbol{r}^2 = u^2t^2\cos^2\alpha + u^2t^2\sin^2\alpha - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$ $= u^2t^2 - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$	M1 A1
			$\frac{d}{dt}(\boldsymbol{r}^2) = 2u^2t - 3ugt^2\sin\alpha + g^2t^3$ $= t(2u^2 - 3ugt\sin\alpha + g^2t^2)$ $= t(2u^2 - \frac{9}{4}u^2\sin^2\alpha + (gt - \frac{3}{2}u\sin\alpha)^2)$	M1 A1 M1 M1
			If $\sin\alpha < \frac{2\sqrt{2}}{3}$, then $2u^2 - \frac{9}{4}u^2\sin^2\alpha > 0$ and distance is always increasing.	M1 A1 (AG)
			If $\sin\alpha > \frac{2\sqrt{2}}{3}$, then distance is decreasing at $t = \frac{3u}{2g}\sin\alpha$ Landing occurs at $t = \frac{2u}{g}\sin\alpha$, which is later (Or imagine falls through ground. Distance increasing while underground, so any decrease must be above ground)	M1 E1 (11 marks)
	(ii)		$\boldsymbol{r} = \begin{pmatrix} ut\cos\alpha + vt \\ ut\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix}$	B1
			$PQ^2 = (u\cos\alpha + v)^2t^2 + u^2t^2\sin^2\alpha - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$	M1 A1
			$\frac{d}{dt}(PQ^2) = 2t(u\cos\alpha + v)^2 + 2u^2t\sin^2\alpha + 2tu^2\sin^2\alpha - 3ugt^2\sin\alpha + g^2t^3$	M1
			$= t\left(2(u\cos\alpha + v)^2 + 2u^2\sin^2\alpha - \frac{9}{4}u^2\sin^2\alpha + \left(gt - \frac{3}{2}u\sin\alpha\right)^2\right)$	M1
			$= t\left(2(u\cos\alpha + v)^2 - \frac{1}{4}u^2\sin^2\alpha + \left(gt - \frac{3}{2}u\sin\alpha\right)^2\right)$	M1 A1
			If $2\sqrt{2}v > (\sin\alpha - 2\sqrt{2}\cos\alpha)u$, then $8(u\cos\alpha + v)^2 > u^2\sin^2\alpha$ So PQ is increasing for all t.	M1 A1 (AG) (9 marks)

10	(i)		Correct diagram Moments about A: $Wa\cos\theta\,(1+2k) = 2aT\sin 2\theta$ If $2k+1 > 4\sin\theta$ then $2T\sin 2\theta > W\cos\theta\,(4\sin\theta) = 2W\sin 2\theta$ Since $\sin 2\theta > 0$, $T > W$ and so the string will break.	B2 M1 M1 A1 A1 (AG) (6 marks)
	(ii)		Resolving forces vertically: $R = ((k+1)W - T\sin\theta)$ Resolving horizontally, ring will slip if: $T\cos\theta > \mu\big((k+1)W - T\sin\theta\big)$ (= max value for friction) Moments about A: $W(2k+1) = 4T\sin\theta$ $\mu\big((k+1)W - T\sin\theta\big) = \mu\left(\frac{4(k+1)}{2k+1} - 1\right)T\sin\theta$ $\mu\left(\frac{2k+3}{2k+1}\right)T\sin\theta$ If $2k+1 > (2k+3)\mu\tan\theta$, then $\mu\left(\frac{2k+3}{2k+1}\right)\sin\theta < \cos\theta$ So the ring will slip.	 M1 M1 M1 A1 M1 A1 (AG) (6 marks)
	(iii)		Attempt to solve breaking inequality for k Breaks at $k = \frac{4\sin\theta - 1}{2}$ Attempt to solve slipping inequality for k Slips at $k = \frac{3\mu\tan\theta - 1}{2(1-\mu\tan\theta)}$ If ring slips before it breaks: $\frac{3\mu\tan\theta - 1}{2(1-\mu\tan\theta)} < \frac{4\sin\theta - 1}{2}$ (for A1, do not allow >) Confirming that inequality is being multiplied by a positive quantity. $3\mu\tan\theta - 1 < 4\sin\theta - 1$ $\mu < \frac{2\cos\theta}{2\sin\theta + 1}$	M1 A1 B1 M1 A1 E1 M1 A1 (AG) (8 marks)

11	(i)		In both cases, award the M mark if all possible values of n_2 for at least 3 values of n_1 are identified. $\boldsymbol{n_3 = 9}$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 8$ $n_1 = 3; n_2 = 8, 7$ $n_1 = 4; n_2 = 8, 7, 6$ $n_1 = 5; n_2 = 8, 7, 6$ $n_1 = 6; n_2 = 8, 7$ $n_1 = 7; n_2 = 8$ $n_1 = 8; n_2$ has no options $\text{Total} = (1 + 2 + 3) \times 2 = 12$ $\boldsymbol{n_3 = 10}$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 9$ $n_1 = 3; n_2 = 9, 8$ $n_1 = 4; n_2 = 9, 8, 7$ $n_1 = 5; n_2 = 9, 8, 7, 6$ $n_1 = 6; n_2 = 9, 8, 7$ $n_1 = 7; n_2 = 9, 8$ $n_1 = 8; n_2 = 9$ $n_1 = 0; n_2$ has no options $\text{Total} = (1 + 2 + 3 + 4) \times 2 - 4 = 16$ $\boldsymbol{n_3 = 2n + 1}$ Total ways $= \big(1 + \cdots + (n-1)\big) \times 2$ (method mark may be implicit) $= (n-1)n$ $\boldsymbol{n_3 = 2n}$ Total ways $= \big(1 + \cdots + (n-1)\big) \times 2 - (n-1)$ (method mark may be implicit) $= (n-1)^2$	 M1 M1 A1 (both totals correct) M1 A1 M1 A1 (7 marks)
	(ii)		Total number of pairs is $\binom{N-1}{2} = \frac{1}{2}(N-1)(N-2)$ Justification for using first part of question $\boldsymbol{N = 2n + 1}$ $\text{Prob} = \frac{(n-1)n}{\frac{1}{2}(2n)(2n-1)} = \frac{n-1}{2n-1}$ $\boldsymbol{N = 2n}$ $\text{Prob} = \frac{(n-1)^2}{\frac{1}{2}(2n-1)(2n-2)} = \frac{n-1}{2n-1}$	 M1 B1 A1 (AG) A1 (4 marks)

	(iii)		$\text{Prob} = \sum_{n=1}^{M} \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n+1) + \sum_{n=1}^{M} \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n)$	M1 A1 (ft)
			$= \sum_{n=1}^{M} \frac{n-1}{2n-1}\left(\frac{\binom{2n}{2}}{\binom{2M+1}{3}} + \frac{\binom{2n-1}{2}}{\binom{2M+1}{3}}\right)$	
			$= \frac{6}{(2M+1)(2M)(2M-1)} \cdot \frac{1}{2}\sum_{n=1}^{M} \frac{n-1}{2n-1}(2n(2n-1) + (2n-1)(2n-2))$ (Use of formula for binomial coefficients with factorials cancelled)	M1 A1
			$= \frac{3}{M(2M+1)(2M-1)} \sum_{n=1}^{M} (n-1)\,(2n-1)$	M1
			Use of $\sum_1^K k^2 = \frac{1}{6}K(K+1)(2K+1)$ to simplify above	M1
			$= \frac{3}{M(2M+1)(2M-1)}\left(\frac{1}{3}M(M+1)(2M+1) - 3 \times \frac{1}{2}M(M+1) + M\right)$	M1
			$= \frac{1}{2(2M+1)(2M-1)}(4M^2 - 3M - 1)$	
			$= \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$	M1 A1
				(9 marks)

12	(i)		$\mu = \int_0^1 nx^n \, dx = \frac{n}{n+1}$	M1 A1
			$\mathbb{E}(X^2) = \int_0^1 nx^{n+1} \, dx = \frac{n}{n+2}$	M1
			$\sigma^2 = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n}{(n+1)^2(n+2)}$	M1 A1 (AG)
				(5 marks)
	(ii)		$\text{LQ} = \frac{1}{2}, \text{UQ} = \frac{\sqrt{3}}{2}$	M1
			$\text{IQR} = \frac{\sqrt{3}-1}{2}$	A1
			$2\sigma = \frac{\sqrt{2}}{3}$	B1
			Squaring IQR and 2σ	M1
			Comparing $\sqrt{3}$ with a rational number…	M1
			…by squaring both sides	M1
			Argument correct	A1
				(7 marks)
	(iii)		$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2}x^2 + \cdots$	A1
			$+ \dfrac{n(n-1)\cdots(n-k+1)}{k!}x^k + \cdots$	A1
			$\text{LQ} = \left(\frac{1}{4}\right)^{1/n}$ and Median $= \left(\frac{1}{2}\right)^{1/n}$	B1
			$\left(\frac{1}{\mu}\right)^n = \left(1 + \frac{1}{n}\right)^n > 1 + n\left(\frac{1}{n}\right) = 2$	M1
			So $\mu < \left(\frac{1}{2}\right)^{1/n}$	A1
			$\left(\dfrac{1}{\mu}\right)^n = \left(1 + \dfrac{1}{n}\right)^n < 1 + n\left(\dfrac{1}{n}\right) + \dfrac{n^2}{2!}\left(\dfrac{1}{n}\right)^2 + \cdots + \dfrac{n^k}{k!}\left(\dfrac{1}{n}\right)^k + \cdots$	M1
			$< 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{k!} + \cdots$	M1
			< 4	
			So $\mu > \left(\frac{1}{4}\right)^{1/n}$	A1
				(8 marks)